3.0 Math Review

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Although this chapter provides a review of some of the mathematical concepts of arithmetic, algebra, and geometry, it is not intended to be a textbook. You should use this chapter to familiarize yourself with the kinds of topics that may be tested in the GMAT® exam. You may wish to consult an arithmetic, algebra, or geometry book for a more detailed discussion of some of the topics.

Section 3.1, "Arithmetic," includes the following topics:

1. Properties of Integers
2. Fractions
3. Decimals
4. Real Numbers
5. Ratio and Proportion
6. Percents
7. Powers and Roots of Numbers
8. Descriptive Statistics
9. Sets
10. Counting Methods
11. Discrete Probability

Section 3.2, "Algebra," does not extend beyond what is usually covered in a first-year high school algebra course. The topics included are as follows:

1. Simplifying Algebraic Expressions
2. Equations
3. Solving Linear Equations with One Unknown
4. Solving Two Linear Equations with Two Unknowns
5. Solving Equations by Factoring
6. Solving Quadratic Equations
7. Exponents
8. Inequalities
9. Absolute Value
10. Functions

Section 3.3, "Geometry," is limited primarily to measurement and intuitive geometry or spatial visualization. Extensive knowledge of theorems and the ability to construct proofs, skills that are usually developed in a formal geometry course, are not tested. The topics included in this section are the following:

1. Lines
2. Intersecting Lines and Angles
3. Perpendicular Lines
4. Parallel Lines
5. Polygons (Convex)
6. Triangles
7. Quadrilaterals
8. Circles
9. Rectangular Solids and Cylinders
10. Coordinate Geometry

Section 3.4, "Word Problems," presents examples of and solutions to the following types of word problems:

1. Rate Problems
2. Work Problems
3. Mixture Problems
4. Interest Problems
5. Discount
6. Profit
7. Sets
8. Geometry Problems
9. Measurement Problems
10. Data Interpretation

3.1 Arithmetic

1. Properties of Integers

An *integer* is any number in the set {. . . –3, –2, –1, 0, 1, 2, 3, . . .}. If x and y are integers and $x \neq 0$, then x is a *divisor* (*factor*) of y provided that $y = xn$ for some integer n. In this case, y is also said to be *divisible* by x or to be a *multiple* of x. For example, 7 is a divisor or factor of 28 since 28 = (7)(4), but 8 is not a divisor of 28 since there is no integer n such that $28 = 8n$.

If x and y are positive integers, there exist unique integers q and r, called the *quotient* and *remainder*, respectively, such that $y = xq + r$ and $0 \leq r < x$. For example, when 28 is divided by 8, the quotient is 3 and the remainder is 4 since 28 = (8)(3) + 4. Note that y is divisible by x if and only if the remainder r is 0; for example, 32 has a remainder of 0 when divided by 8 because 32 is divisible by 8. Also, note that when a smaller integer is divided by a larger integer, the quotient is 0 and the remainder is the smaller integer. For example, 5 divided by 7 has the quotient 0 and the remainder 5 since 5 = (7)(0) + 5.

Any integer that is divisible by 2 is an *even integer*; the set of even integers is {. . . –4, –2, 0, 2, 4, 6, 8, . . .}. Integers that are not divisible by 2 are *odd integers*; {. . . –3, –1, 1, 3, 5, . . .} is the set of odd integers.

If at least one factor of a product of integers is even, then the product is even; otherwise the product is odd. If two integers are both even or both odd, then their sum and their difference are even. Otherwise, their sum and their difference are odd.

A *prime* number is a positive integer that has exactly two different positive divisors, 1 and itself. For example, 2, 3, 5, 7, 11, and 13 are prime numbers, but 15 is not, since 15 has four different positive divisors, 1, 3, 5, and 15. The number 1 is not a prime number since it has only one positive divisor. Every integer greater than 1 either is prime or can be uniquely expressed as a product of prime factors. For example, 14 = (2)(7), 81 = (3)(3)(3)(3), and 484 = (2)(2)(11)(11).

The numbers –2, –1, 0, 1, 2, 3, 4, 5 are *consecutive integers.* Consecutive integers can be represented by $n, n + 1, n + 2, n + 3, \ldots$, where n is an integer. The numbers 0, 2, 4, 6, 8 are *consecutive even integers,* and 1, 3, 5, 7, 9 are *consecutive odd integers.* Consecutive even integers can be represented by $2n, 2n + 2, 2n + 4, \ldots$, and consecutive odd integers can be represented by $2n + 1, 2n + 3, 2n + 5, \ldots$, where n is an integer.

Properties of the integer 1. If n is any number, then $1 \cdot n = n$, and for any number $n \neq 0$, $n \cdot \frac{1}{n} = 1$.

The number 1 can be expressed in many ways; for example, $\frac{n}{n} = 1$ for any number $n \neq 0$.

Multiplying or dividing an expression by 1, in any form, does not change the value of that expression.

Properties of the integer 0. The integer 0 is neither positive nor negative. If n is any number, then $n + 0 = n$ and $n \cdot 0 = 0$. Division by 0 is not defined.

2. Fractions

In a fraction $\frac{n}{d}$, n is the *numerator* and d is the *denominator.* The denominator of a fraction can never be 0, because division by 0 is not defined.

Two fractions are said to be *equivalent* if they represent the same number. For example, $\frac{8}{36}$ and $\frac{14}{63}$ are equivalent since they both represent the number $\frac{2}{9}$. In each case, the fraction is reduced to lowest terms by dividing both numerator and denominator by their *greatest common divisor* (gcd). The gcd of 8 and 36 is 4 and the gcd of 14 and 63 is 7.

Addition and subtraction of fractions.

Two fractions with the same denominator can be added or subtracted by performing the required operation with the numerators, leaving the denominators the same. For example, $\frac{3}{5}+\frac{4}{5}=\frac{3+4}{5}=\frac{7}{5}$ and $\frac{5}{7}-\frac{2}{7}=\frac{5-2}{7}=\frac{3}{7}$. If two fractions do not have the same denominator, express them as equivalent fractions with the same denominator. For example, to add $\frac{3}{5}$ and $\frac{4}{7}$, multiply the numerator and denominator of the first fraction by 7 and the numerator and denominator of the second fraction by 5, obtaining $\frac{21}{35}$ and $\frac{20}{35}$, respectively; $\frac{21}{35}+\frac{20}{35}=\frac{41}{35}$.

For the new denominator, choosing the *least common multiple* (lcm) of the denominators usually lessens the work. For $\frac{2}{3}+\frac{1}{6}$, the lcm of 3 and 6 is 6 (not $3 \times 6 = 18$), so $\frac{2}{3}+\frac{1}{6}=\frac{2}{3}\times\frac{2}{2}+\frac{1}{6}=\frac{4}{6}+\frac{1}{6}=\frac{5}{6}$.

Multiplication and division of fractions.

To multiply two fractions, simply multiply the two numerators and multiply the two denominators.

For example, $\frac{2}{3}\times\frac{4}{7}=\frac{2\times 4}{3\times 7}=\frac{8}{21}$.

To divide by a fraction, invert the divisor (that is, find its *reciprocal*) and multiply. For example, $\frac{2}{3}\div\frac{4}{7}=\frac{2}{3}\times\frac{7}{4}=\frac{14}{12}=\frac{7}{6}$.

In the problem above, the reciprocal of $\frac{4}{7}$ is $\frac{7}{4}$. In general, the reciprocal of a fraction $\frac{n}{d}$ is $\frac{d}{n}$, where n and d are not zero.

Mixed numbers.

A number that consists of a whole number and a fraction, for example, $7\frac{2}{3}$, is a mixed number: $7\frac{2}{3}$ means $7+\frac{2}{3}$.

To change a mixed number into a fraction, multiply the whole number by the denominator of the fraction and add this number to the numerator of the fraction; then put the result over the denominator of the fraction. For example, $7\frac{2}{3}=\frac{(3\times 7)+2}{3}=\frac{23}{3}$.

3. Decimals

In the decimal system, the position of the period or *decimal point* determines the place value of the digits. For example, the digits in the number 7,654.321 have the following place values:

Thousands		Hundreds	Tens	Ones or units		Tenths	Hundredths	Thousandths
7	,	6	5	4	.	3	2	1

Some examples of decimals follow.

$$0.321 = \frac{3}{10} + \frac{2}{100} + \frac{1}{1,000} = \frac{321}{1,000}$$

$$0.0321 = \frac{0}{10} + \frac{3}{100} + \frac{2}{1,000} + \frac{1}{10,000} = \frac{321}{10,000}$$

$$1.56 = 1 + \frac{5}{10} + \frac{6}{100} = \frac{156}{100}$$

Sometimes decimals are expressed as the product of a number with only one digit to the left of the decimal point and a power of 10. This is called *scientific notation.* For example, 231 can be written as 2.31×10^2 and 0.0231 can be written as 2.31×10^{-2}. When a number is expressed in scientific notation, the exponent of the 10 indicates the number of places that the decimal point is to be moved in the number that is to be multiplied by a power of 10 in order to obtain the product. The decimal point is moved to the right if the exponent is positive and to the left if the exponent is negative. For example, 2.013×10^4 is equal to 20,130 and 1.91×10^{-4} is equal to 0.000191.

Addition and subtraction of decimals.

To add or subtract two decimals, the decimal points of both numbers should be lined up. If one of the numbers has fewer digits to the right of the decimal point than the other, zeros may be inserted to the right of the last digit. For example, to add 17.6512 and 653.27, set up the numbers in a column and add:

$$\begin{array}{r} 17.6512 \\ +\,653.2700 \\ \hline 670.9212 \end{array}$$

Likewise for 653.27 minus 17.6512:

$$\begin{array}{r} 653.2700 \\ -17.6512 \\ \hline 635.6188 \end{array}$$

Multiplication of decimals.

To multiply decimals, multiply the numbers as if they were whole numbers and then insert the decimal point in the product so that the number of digits to the right of the decimal point is equal to the sum of the numbers of digits to the right of the decimal points in the numbers being multiplied. For example:

$$\begin{array}{rl} 2.09 & \text{(2 digits to the right)} \\ \times 1.3 & \text{(1 digit to the right)} \\ \hline 627 & \\ 2090 & \\ \hline 2.717 & (2+1=3 \text{ digits to the right}) \end{array}$$

Division of decimals.

To divide a number (the dividend) by a decimal (the divisor), move the decimal point of the divisor to the right until the divisor is a whole number. Then move the decimal point of the dividend the same number of places to the right, and divide as you would by a whole number. The decimal point in the quotient will be directly above the decimal point in the new dividend. For example, to divide 698.12 by 12.4:

$$12.4\overline{)698.12}$$

will be replaced by:

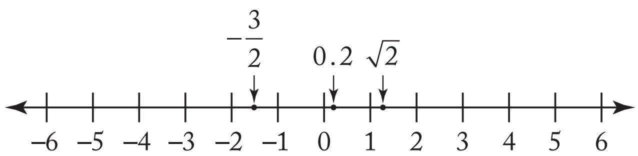
$$124\overline{)6981.2}$$

and the division would proceed as follows:

$$\begin{array}{r} 56.3 \\ 124\overline{)6981.2} \\ \underline{620} \\ 781 \\ \underline{744} \\ 37\,2 \\ \underline{37\,2} \\ 0 \end{array}$$

4. Real Numbers

All *real* numbers correspond to points on the number line and all points on the number line correspond to real numbers. All real numbers except zero are either positive or negative.



On a number line, numbers corresponding to points to the left of zero are negative and numbers corresponding to points to the right of zero are positive. For any two numbers on the number line, the number to the left is less than the number to the right; for example, $-4 < -3 < -\frac{3}{2} < -1$, and $1 < \sqrt{2} < 2$.

To say that the number n is between 1 and 4 on the number line means that $n > 1$ and $n < 4$, that is, $1 < n < 4$. If n is "between 1 and 4, inclusive," then $1 \le n \le 4$.

The distance between a number and zero on the number line is called the *absolute value* of the number. Thus 3 and −3 have the same absolute value, 3, since they are both three units from zero. The absolute value of 3 is denoted $|3|$. Examples of absolute values of numbers are

$$|-5| = |5| = 5, \left|-\frac{7}{2}\right| = \frac{7}{2}, \text{ and } |0| = 0.$$

Note that the absolute value of any nonzero number is positive.

Here are some properties of real numbers that are used frequently. If x, y, and z are real numbers, then

(1) $x + y = y + x$ and $xy = yx$.
For example, $8 + 3 = 3 + 8 = 11$, and $(17)(5) = (5)(17) = 85$.

(2) $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$.
For example, $(7 + 5) + 2 = 7 + (5 + 2) = 7 + (7) = 14$, and $\left(5\sqrt{3}\right)\left(\sqrt{3}\right) = (5)\left(\sqrt{3}\sqrt{3}\right) = (5)(3) = 15$.

(3) $xy + xz = x(y + z)$.
For example, $718(36) + 718(64) = 718(36 + 64) = 718(100) = 71{,}800$.

(4) If x and y are both positive, then $x + y$ and xy are positive.

(5) If x and y are both negative, then $x + y$ is negative and xy is positive.

(6) If x is positive and y is negative, then xy is negative.

(7) If $xy = 0$, then $x = 0$ or $y = 0$. For example, $3y = 0$ implies $y = 0$.

(8) $|x + y| \le |x| + |y|$. For example, if $x = 10$ and $y = 2$, then $|x + y| = |12| = 12 = |x| + |y|$; and if $x = 10$ and $y = -2$, then $|x + y| = |8| = 8 < 12 = |x| + |y|$.

5. Ratio and Proportion

The *ratio* of the number a to the number b ($b \ne 0$) is $\frac{a}{b}$.

A ratio may be expressed or represented in several ways. For example, the ratio of 2 to 3 can be written as 2 to 3, 2:3, or $\frac{2}{3}$. The order of the terms of a ratio is important. For example, the ratio of the number of months with exactly 30 days to the number with exactly 31 days is $\frac{4}{7}$, not $\frac{7}{4}$.

A *proportion* is a statement that two ratios are equal; for example, $\frac{2}{3} = \frac{8}{12}$ is a proportion. One way to solve a proportion involving an unknown is to cross multiply, obtaining a new equality. For example, to solve for n in the proportion $\frac{2}{3} = \frac{n}{12}$, cross multiply, obtaining $24 = 3n$; then divide both sides by 3, to get $n = 8$.

6. Percents

Percent means *per hundred* or *number out of 100.* A percent can be represented as a fraction with a denominator of 100, or as a decimal. For example:

$$37\% = \frac{37}{100} = 0.37.$$

To find a certain percent of a number, multiply the number by the percent expressed as a decimal or fraction. For example:

$$20\% \text{ of } 90 = 0.2 \times 90 = 18$$

or

$$20\% \text{ of } 90 = \frac{20}{100} \times 90 = \frac{1}{5} \times 90 = 18.$$

Percents greater than 100%.

Percents greater than 100% are represented by numbers greater than 1. For example:

$$300\% = \frac{300}{100} = 3$$
$$250\% \text{ of } 80 = 2.5 \times 80 = 200.$$

Percents less than 1%.

The percent 0.5% means $\frac{1}{2}$ of 1 percent. For example, 0.5% of 12 is equal to $0.005 \times 12 = 0.06$.

Percent change.

Often a problem will ask for the percent increase or decrease from one quantity to another quantity. For example, "If the price of an item increases from \$24 to \$30, what is the percent increase in price?" To find the percent increase, first find the amount of the increase; then divide this increase by the original amount, and express this quotient as a percent. In the example above, the percent increase would be found in the following way: the amount of the increase is $(30 - 24) = 6$. Therefore, the percent increase is $\frac{6}{24} = 0.25 = 25\%$.

Likewise, to find the percent decrease (for example, the price of an item is reduced from \$30 to \$24), first find the amount of the decrease; then divide this decrease by the original amount, and express this quotient as a percent. In the example above, the amount of decrease is $(30 - 24) = 6$.

Therefore, the percent decrease is $\frac{6}{30} = 0.20 = 20\%$.

Note that the percent increase from 24 to 30 is not the same as the percent decrease from 30 to 24.

In the following example, the increase is greater than 100 percent: If the cost of a certain house in 1983 was 300 percent of its cost in 1970, by what percent did the cost increase?

If n is the cost in 1970, then the percent increase is equal to $\frac{3n-n}{n} = \frac{2n}{n} = 2$, or 200%.

7. Powers and Roots of Numbers

When a number k is to be used n times as a factor in a product, it can be expressed as k^n, which means the nth power of k. For example, $2^2 = 2 \times 2 = 4$ and $2^3 = 2 \times 2 \times 2 = 8$ are powers of 2.

Squaring a number that is greater than 1, or raising it to a higher power, results in a larger number; squaring a number between 0 and 1 results in a smaller number. For example:

$$3^2 = 9 \qquad (9 > 3)$$
$$\left(\frac{1}{3}\right)^2 = \frac{1}{9} \qquad \left(\frac{1}{9} < \frac{1}{3}\right)$$
$$(0.1)^2 = 0.01 \qquad (0.01 < 0.1)$$

A *square root* of a number n is a number that, when squared, is equal to n. The square root of a negative number is not a real number. Every positive number n has two square roots, one positive and the other negative, but $\sqrt{n}$ denotes the positive number whose square is n. For example, $\sqrt{9}$ denotes 3. The two square roots of 9 are $\sqrt{9} = 3$ and $-\sqrt{9} = -3$.

Every real number r has exactly one real *cube root,* which is the number s such that $s^3 = r$. The real cube root of r is denoted by $\sqrt[3]{r}$. Since $2^3 = 8$, $\sqrt[3]{8} = 2$. Similarly, $\sqrt[3]{-8} = -2$, because $(-2)^3 = -8$.

8. Descriptive Statistics

A list of numbers, or numerical data, can be described by various statistical measures. One of the most common of these measures is the *average,* or *(arithmetic) mean,* which locates a type of "center" for the data. The average of n numbers is defined as the sum of the n numbers divided by n. For example, the average of 6, 4, 7, 10, and 4 is $\frac{6+4+7+10+4}{5} = \frac{31}{5} = 6.2$.

The *median* is another type of center for a list of numbers. To calculate the median of n numbers, first order the numbers from least to greatest; if n is odd, the median is defined as the middle number, whereas if n is even, the median is defined as the average of the two middle numbers. In the example above, the numbers, in order, are 4, 4, 6, 7, 10, and the median is 6, the middle number.

For the numbers 4, 6, 6, 8, 9, 12, the median is $\frac{6+8}{2} = 7$. Note that the mean of these numbers is 7.5.

The median of a set of data can be less than, equal to, or greater than the mean. Note that for a large set of data (for example, the salaries of 800 company employees), it is often true that about half of the data is less than the median and about half of the data is greater than the median; but this is not always the case, as the following data show.

3, 5, 7, 7, 7, 7, 7, 7, 8, 9, 9, 9, 9, 10, 10

Here the median is 7, but only $\frac{2}{15}$ of the data is less than the median.

The *mode* of a list of numbers is the number that occurs most frequently in the list. For example, the mode of 1, 3, 6, 4, 3, 5 is 3. A list of numbers may have more than one mode. For example, the list 1, 2, 3, 3, 3, 5, 7, 10, 10, 10, 20 has two modes, 3 and 10.

The degree to which numerical data are spread out or dispersed can be measured in many ways. The simplest measure of dispersion is the *range,* which is defined as the greatest value in the numerical data minus the least value. For example, the range of 11, 10, 5, 13, 21 is $21 - 5 = 16$. Note how the range depends on only two values in the data.

One of the most common measures of dispersion is the *standard deviation.* Generally speaking, the more the data are spread away from the mean, the greater the standard deviation. The standard deviation of n numbers can be calculated as follows: (1) find the arithmetic mean, (2) find the differences between the mean and each of the n numbers, (3) square each of the differences, (4) find the average of the squared differences, and (5) take the nonnegative square root of this average. Shown below is this calculation for the data 0, 7, 8, 10, 10, which have arithmetic mean 7.

x	$x-7$	$(x-7)^2$
0	−7	49
7	0	0
8	1	1
10	3	9
10	3	9
	Total	68

Standard deviation $\sqrt{\frac{68}{5}} \approx 3.7$

Notice that the standard deviation depends on every data value, although it depends most on values that are farthest from the mean. This is why a distribution with data grouped closely around the mean will have a smaller standard deviation than will data spread far from the mean. To illustrate this, compare the data 6, 6, 6.5, 7.5, 9, which also have mean 7. Note that the numbers in the second set of data seem to be grouped more closely around the mean of 7 than the numbers in the first set. This is reflected in the standard deviation, which is less for the second set (approximately 1.1) than for the first set (approximately 3.7).

There are many ways to display numerical data that show how the data are distributed. One simple way is with a *frequency distribution,* which is useful for data that have values occurring with varying frequencies. For example, the 20 numbers

–4	0	0	–3	–2	–1	–1	0	–1	–4
–1	–5	0	–2	0	–5	–2	0	0	–1

are displayed on the next page in a frequency distribution by listing each different value x and the frequency f with which x occurs.

Data Value x	Frequency f
–5	2
–4	2
–3	1
–2	3
–1	5
0	7
Total	20

From the frequency distribution, one can readily compute descriptive statistics:

Mean: $= \dfrac{(-5)(2)+(-4)(2)+(-3)(1)+(-2)(3)+(-1)(5)+(0)(7)}{20} = -1.6$

Median: –1 (the average of the 10th and 11th numbers)

Mode: 0 (the number that occurs most frequently)

Range: 0 – (–5) = 5

Standard deviation: $\sqrt{\dfrac{(-5+1.6)^2(2)+(-4+1.6)^2(2)+\ldots+(0+1.6)^2(7)}{20}} \approx 1.7$

9. Sets

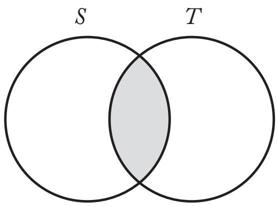
In mathematics a *set* is a collection of numbers or other objects. The objects are called the *elements* of the set. If S is a set having a finite number of elements, then the number of elements is denoted by $|S|$. Such a set is often defined by listing its elements; for example, $S = \{-5, 0, 1\}$ is a set with $|S| = 3$.

The order in which the elements are listed in a set does not matter; thus $\{-5, 0, 1\} = \{0, 1, -5\}$.

If all the elements of a set S are also elements of a set T, then S is a *subset* of T; for example, $S = \{-5, 0, 1\}$ is a subset of $T = \{-5, 0, 1, 4, 10\}$.

For any two sets A and B, the *union* of A and B is the set of all elements that are in A *or* in B *or* in both. The *intersection* of A and B is the set of all elements that are both in A *and* in B. The union is denoted by $A \cup B$ and the intersection is denoted by $A \cap B$. As an example, if $A = \{3, 4\}$ and $B = \{4, 5, 6\}$, then $A \cup B = \{3, 4, 5, 6\}$ and $A \cap B = \{4\}$. Two sets that have no elements in common are said to be *disjoint* or *mutually exclusive.*

The relationship between sets is often illustrated with a *Venn diagram* in which sets are represented by regions in a plane. For two sets S and T that are not disjoint and neither is a subset of the other, the intersection $S \cap T$ is represented by the shaded region of the diagram below.



This diagram illustrates a fact about any two finite sets S and T: the number of elements in their union equals the sum of their individual numbers of elements minus the number of elements in their intersection (because the latter are counted twice in the sum); more concisely,

$$|S \cup T| = |S| + |T| - |S \cap T|.$$

This counting method is called the general addition rule for two sets. As a special case, if S and T are disjoint, then

$$|S \cup T| = |S| + |T|$$

since $|S \cap T| = 0$.

10. Counting Methods

There are some useful methods for counting objects and sets of objects without actually listing the elements to be counted. The following principle of multiplication is fundamental to these methods.

If an object is to be chosen from a set of m objects and a second object is to be chosen from a different set of n objects, then there are mn ways of choosing both objects simultaneously.

As an example, suppose the objects are items on a menu. If a meal consists of one entree and one dessert and there are 5 entrees and 3 desserts on the menu, then there are $5 \times 3 = 15$ different meals that can be ordered from the menu. As another example, each time a coin is flipped, there are two possible outcomes, heads and tails. If an experiment consists of 8 consecutive coin flips, then the experiment has 2^8 possible outcomes, where each of these outcomes is a list of heads and tails in some order.

A symbol that is often used with the multiplication principle is the *factorial.* If n is an integer greater than 1, then n factorial, denoted by the symbol $n!$, is defined as the product of all the integers from 1 to n. Therefore,

$$2! = (1)(2) = 2,$$
$$3! = (1)(2)(3) = 6,$$
$$4! = (1)(2)(3)(4) = 24, \text{ etc.}$$

Also, by definition, $0! = 1! = 1$.

The factorial is useful for counting the number of ways that a set of objects can be ordered. If a set of n objects is to be ordered from 1st to nth, then there are n choices for the 1st object, $n - 1$ choices for the 2nd object, $n - 2$ choices for the 3rd object, and so on, until there is only 1 choice for the nth object. Thus, by the multiplication principle, the number of ways of ordering the n objects is

$$n(n-1)(n-2)\cdots(3)(2)(1) = n!.$$

For example, the number of ways of ordering the letters A, B, and C is 3!, or 6:

ABC, ACB, BAC, BCA, CAB, and CBA.

These orderings are called the *permutations* of the letters A, B, and C.

A permutation can be thought of as a selection process in which objects are selected one by one in a certain order. If the order of selection is not relevant and only k objects are to be selected from a larger set of n objects, a different counting method is employed.

Specifically, consider a set of n objects from which a complete selection of k objects is to be made without regard to order, where $0 \le k \le n$. Then the number of possible complete selections of k objects is called the number of *combinations* of n objects taken k at a time and is denoted by $\binom{n}{k}$.

The value of $\binom{n}{k}$ is given by $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Note that $\binom{n}{k}$ is the number of k-element subsets of a set with n elements. For example, if $S = \{A, B, C, D, E\}$, then the number of 2-element subsets of S, or the number of combinations of 5 letters taken 2 at a time, is $\binom{5}{2} = \frac{5!}{2!3!} = \frac{120}{(2)(6)} = 10$.

The subsets are {A, B}, {A, C}, {A, D}, {A, E}, {B, C}, {B, D}, {B, E}, {C, D}, {C, E}, and {D, E}. Note that $\binom{5}{2} = 10 = \binom{5}{3}$ because every 2-element subset chosen from a set of 5 elements corresponds to a unique 3-element subset consisting of the elements *not* chosen.

In general, $\binom{n}{k} = \binom{n}{n-k}$.

11. Discrete Probability

Many of the ideas discussed in the preceding three topics are important to the study of discrete probability. Discrete probability is concerned with *experiments* that have a finite number of *outcomes*. Given such an experiment, an *event* is a particular set of outcomes. For example, rolling a number cube with faces numbered 1 to 6 (similar to a 6-sided die) is an experiment with 6 possible outcomes: 1, 2, 3, 4, 5, or 6. One event in this experiment is that the outcome is 4, denoted {4}; another event is that the outcome is an odd number: {1, 3, 5}.

The probability that an event E occurs, denoted by $P(E)$, is a number between 0 and 1, inclusive. If E has no outcomes, then E is *impossible* and $P(E) = 0$; if E is the set of all possible outcomes of the experiment, then E is *certain* to occur and $P(E) = 1$. Otherwise, E is possible but uncertain, and $0 < P(E) < 1$. If F is a subset of E, then $P(F) \le P(E)$. In the example above, if the probability of each of the 6 outcomes is the same, then the probability of each outcome is $\frac{1}{6}$, and the outcomes are said to be *equally likely*. For experiments in which all the individual outcomes are equally likely, the probability of an event E is

$$P(E) = \frac{\text{The number of outcomes in } E}{\text{The total number of possible outcomes}}.$$

In the example, the probability that the outcome is an odd number is

$$P(\{1,3,5\}) = \frac{|\{1,3,5\}|}{6} = \frac{3}{6} = \frac{1}{2}.$$

Given an experiment with events E and F, the following events are defined:
"not E" is the set of outcomes that are not outcomes in E;
"E or F" is the set of outcomes in E or F or both, that is, $E \cup F$;
"E and F" is the set of outcomes in both E and F, that is, $E \cap F$.

The probability that E does not occur is $P(\text{not } E) = 1 - P(E)$. The probability that "$E$ or F" occurs is $P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$, using the general addition rule at the end of section 3.1.9 ("Sets"). For the number cube, if E is the event that the outcome is an odd number, {1, 3, 5}, and F is the event that the outcome is a prime number, {2, 3, 5}, then $P(E \text{ and } F) = P(\{3,5\}) = \frac{2}{6} = \frac{1}{3}$ and so $P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F) = \frac{3}{6} + \frac{3}{6} - \frac{2}{6} = \frac{4}{6} = \frac{2}{3}$.
Note that the event "E or F" is $E \cup F = \{1, 2, 3, 5\}$, and hence $P(E \text{ or } F) = \frac{|\{1,2,3,5\}|}{6} = \frac{4}{6} = \frac{2}{3}$.

If the event "E and F" is impossible (that is, $E \cap F$ has no outcomes), then E and F are said to be *mutually exclusive* events, and $P(E \text{ and } F) = 0$. Then the general addition rule is reduced to $P(E \text{ or } F) = P(E) + P(F)$.

This is the special addition rule for the probability of two mutually exclusive events.

Two events A and B are said to be *independent* if the occurrence of either event does not alter the probability that the other event occurs. For one roll of the number cube, let $A = \{2, 4, 6\}$ and let $B = \{5, 6\}$. Then the probability that A occurs is $P(A) = \frac{|A|}{6} = \frac{3}{6} = \frac{1}{2}$, while, *presuming B occurs,* the probability that A occurs is

$$\frac{|A \cap B|}{|B|} = \frac{|\{6\}|}{|\{5,6\}|} = \frac{1}{2}.$$

Similarly, the probability that B occurs is $P(B) = \frac{|B|}{6} = \frac{2}{6} = \frac{1}{3}$, while, *presuming A occurs,* the probability that B occurs is

$$\frac{|B \cap A|}{|A|} = \frac{|\{6\}|}{|\{2,4,6\}|} = \frac{1}{3}.$$

Thus, the occurrence of either event does not affect the probability that the other event occurs. Therefore, A and B are independent.

The following multiplication rule holds for any independent events E and F: $P(E \text{ and } F) = P(E)P(F)$.

For the independent events A and B above, $P(A \text{ and } B) = P(A)P(B) = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \left(\frac{1}{6}\right)$.

Note that the event "A and B" is $A \cap B = \{6\}$, and hence $P(A \text{ and } B) = P(\{6\}) = \frac{1}{6}$. It follows from the general addition rule and the multiplication rule above that if E and F are independent, then

$$P(E \text{ or } F) = P(E) + P(F) - P(E)P(F).$$

For a final example of some of these rules, consider an experiment with events A, B, and C for which $P(A) = 0.23$, $P(B) = 0.40$, and $P(C) = 0.85$. Also, suppose that events A and B are mutually exclusive and events B and C are independent. Then

$$\begin{aligned} P(A \text{ or } B) &= P(A) + P(B) \text{ (since } A \text{ or } B \text{ are mutually exclusive)} \\ &= 0.23 + 0.40 \\ &= 0.63 \\ P(B \text{ or } C) &= P(B) + P(C) - P(B)P(C) \text{ (by independence)} \\ &= 0.40 + 0.85 - (0.40)(0.85) \\ &= 0.91 \end{aligned}$$

Note that $P(A \text{ or } C)$ and $P(A \text{ and } C)$ cannot be determined using the information given. But it can be determined that A and C are *not* mutually exclusive since $P(A) + P(C) = 1.08$, which is greater than 1, and therefore cannot equal $P(A \text{ or } C)$; from this it follows that $P(A \text{ and } C) \geq 0.08$. One can also deduce that $P(A \text{ and } C) \leq P(A) = 0.23$, since $A \cap C$ is a subset of A, and that $P(A \text{ or } C) \geq P(C) = 0.85$ since C is a subset of $A \cup C$. Thus, one can conclude that $0.85 \leq P(A \text{ or } C) \leq 1$ and $0.08 \leq P(A \text{ and } C) \leq 0.23$.

3.2 Algebra

Algebra is based on the operations of arithmetic and on the concept of an *unknown quantity,* or *variable.* Letters such as x or n are used to represent unknown quantities. For example, suppose Pam has 5 more pencils than Fred. If F represents the number of pencils that Fred has, then the number of pencils that Pam has is $F + 5$. As another example, if Jim's present salary S is increased by 7%, then his new salary is $1.07S$. A combination of letters and arithmetic operations, such as $F + 5$, $\frac{3x^2}{2x-5}$, and $19x^2 - 6x + 3$, is called an *algebraic expression.*

The expression $19x^2 - 6x + 3$ consists of the *terms* $19x^2$, $-6x$, and 3, where 19 is the *coefficient* of x^2, -6 is the coefficient of x^1, and 3 is a *constant term* (or coefficient of $x^0 = 1$). Such an expression is called a *second degree* (or *quadratic*) *polynomial in x* since the highest power of x is 2. The expression $F + 5$ is a *first degree* (or *linear*) *polynomial in F* since the highest power of F is 1. The expression $\frac{3x^2}{2x-5}$ is not a polynomial because it is not a sum of terms that are each powers of x multiplied by coefficients.

1. Simplifying Algebraic Expressions

Often when working with algebraic expressions, it is necessary to simplify them by factoring or combining *like* terms. For example, the expression $6x + 5x$ is equivalent to $(6 + 5)x$, or $11x$.
In the expression $9x - 3y$, 3 is a factor common to both terms: $9x - 3y = 3(3x - y)$. In the expression $5x^2 + 6y$, there are no like terms and no common factors.

If there are common factors in the numerator and denominator of an expression, they can be divided out, provided that they are not equal to zero.

For example, if $x \neq 3$, then $\frac{x-3}{x-3}$ is equal to 1; therefore,

$$\begin{aligned}\frac{3xy-9y}{x-3} &= \frac{3y(x-3)}{x-3}\\ &= (3y)(1)\\ &= 3y\end{aligned}$$

To multiply two algebraic expressions, each term of one expression is multiplied by each term of the other expression. For example:

$$\begin{aligned}(3x-4)(9y+x) &= 3x(9y+x)-4(9y+x)\\ &= (3x)(9y)+(3x)(x)+(-4)(9y)+(-4)(x)\\ &= 27xy+3x^2-36y-4x\end{aligned}$$

An algebraic expression can be evaluated by substituting values of the unknowns in the expression. For example, if $x = 3$ and $y = -2$, then $3xy - x^2 + y$ can be evaluated as

$$3(3)(-2)-(3)^2+(-2) = -18-9-2 = -29$$

2. Equations

A major focus of algebra is to solve equations involving algebraic expressions. Some examples of such equations are

$5x - 2 = 9 - x$ (a linear equation with one unknown)

$3x + 1 = y - 2$ (a linear equation with two unknowns)

$5x^2 + 3x - 2 = 7x$ (a quadratic equation with one unknown)

$\frac{x(x-3)(x^2+5)}{x-4} = 0$ (an equation that is factored on one side with 0 on the other)

The *solutions* of an equation with one or more unknowns are those values that make the equation true, or "satisfy the equation," when they are substituted for the unknowns of the equation. An equation may have no solution or one or more solutions. If two or more equations are to be solved together, the solutions must satisfy all the equations simultaneously.

Two equations having the same solution(s) are *equivalent equations.* For example, the equations

$$2 + x = 3$$
$$4 + 2x = 6$$

each have the unique solution $x = 1$. Note that the second equation is the first equation multiplied by 2. Similarly, the equations

$$3x - y = 6$$
$$6x - 2y = 12$$

have the same solutions, although in this case each equation has infinitely many solutions. If any value is assigned to x, then $3x - 6$ is a corresponding value for y that will satisfy both equations; for example, $x = 2$ and $y = 0$ is a solution to both equations, as is $x = 5$ and $y = 9$.

3. Solving Linear Equations with One Unknown

To solve a linear equation with one unknown (that is, to find the value of the unknown that satisfies the equation), the unknown should be isolated on one side of the equation. This can be done by performing the same mathematical operations on both sides of the equation. Remember that if the same number is added to or subtracted from both sides of the equation, this does not change the equality; likewise, multiplying or dividing both sides by the same nonzero number does not change the equality. For example, to solve the equation $\frac{5x-6}{3} = 4$ for x, the variable x can be isolated using the following steps:

$$5x - 6 = 12 \quad \text{(multiplying by 3)}$$
$$5x = 18 \quad \text{(adding 6)}$$
$$x = \frac{18}{5} \quad \text{(dividing by 5)}$$

The solution, $\frac{18}{5}$, can be checked by substituting it for x in the original equation to determine whether it satisfies that equation:

$$\frac{5\left(\frac{18}{5}\right) - 6}{3} = \frac{18-6}{3} = \frac{12}{3} = 4$$

Therefore, $x = \frac{18}{5}$ is the solution.

4. Solving Two Linear Equations with Two Unknowns

For two linear equations with two unknowns, if the equations are equivalent, then there are infinitely many solutions to the equations, as illustrated at the end of section 3.2.2 ("Equations"). If the equations are not equivalent, then they have either one unique solution or no solution. The latter case is illustrated by the two equations:

$$3x+4y=17$$
$$6x+8y=35$$

Note that $3x + 4y = 17$ implies $6x + 8y = 34$, which contradicts the second equation. Thus, no values of x and y can simultaneously satisfy both equations.

There are several methods of solving two linear equations with two unknowns. With any method, if a contradiction is reached, then the equations have no solution; if a trivial equation such as $0 = 0$ is reached, then the equations are equivalent and have infinitely many solutions. Otherwise, a unique solution can be found.

One way to solve for the two unknowns is to express one of the unknowns in terms of the other using one of the equations, and then substitute the expression into the remaining equation to obtain an equation with one unknown. This equation can be solved and the value of the unknown substituted into either of the original equations to find the value of the other unknown. For example, the following two equations can be solved for x and y.

$$(1)\quad 3x+2y=11$$
$$(2)\quad x-y=2$$

In equation (2), $x = 2 + y$. Substitute $2 + y$ in equation (1) for x:

$$3(2+y)+2y=11$$
$$6+3y+2y=11$$
$$6+5y=11$$
$$5y=5$$
$$y=1$$

If $y = 1$, then $x - 1 = 2$ and $x = 2 + 1 = 3$.

There is another way to solve for x and y by eliminating one of the unknowns. This can be done by making the coefficients of one of the unknowns the same (disregarding the sign) in both equations and either adding the equations or subtracting one equation from the other. For example, to solve the equations

$$(1)\quad 6x+5y=29$$
$$(2)\quad 4x-3y=-6$$

by this method, multiply equation (1) by 3 and equation (2) by 5 to get

$$18x+15y=87$$
$$20x-15y=-30$$

Adding the two equations eliminates y, yielding $38x = 57$, or $x = \frac{3}{2}$. Finally, substituting $\frac{3}{2}$ for x in one of the equations gives $y = 4$. These answers can be checked by substituting both values into both of the original equations.

5. Solving Equations by Factoring

Some equations can be solved by factoring. To do this, first add or subtract expressions to bring all the expressions to one side of the equation, with 0 on the other side. Then try to factor the nonzero side into a product of expressions. If this is possible, then using property (7) in section 3.1.4 ("Real Numbers") each of the factors can be set equal to 0, yielding several simpler equations that possibly can be solved. The solutions of the simpler equations will be solutions of the factored equation. As an example, consider the equation $x^3 - 2x^2 + x = -5(x - 1)^2$:

$$\begin{aligned} x^3 - 2x^2 + x + 5(x-1)^2 &= 0 \\ x\left(x^2 - 2x + 1\right) + 5(x-1)^2 &= 0 \\ x(x-1)^2 + 5(x-1)^2 &= 0 \\ (x+5)(x-1)^2 &= 0 \\ x+5 = 0 \text{ or } (x-1)^2 &= 0 \\ x = -5 \text{ or } x &= 1. \end{aligned}$$

For another example, consider $\frac{x(x-3)\left(x^2+5\right)}{x-4} = 0.$ A fraction equals 0 if and only if its numerator equals 0. Thus, $x(x - 3)(x^2 + 5) = 0$:

$$\begin{aligned} &x = 0 \text{ or } x - 3 = 0 \text{ or } x^2 + 5 = 0 \\ &x = 0 \text{ or } x = 3 \text{ or } x^2 + 5 = 0. \end{aligned}$$

But $x^2 + 5 = 0$ has no real solution because $x^2 + 5 > 0$ for every real number. Thus, the solutions are 0 and 3.

The solutions of an equation are also called the *roots* of the equation. These roots can be checked by substituting them into the original equation to determine whether they satisfy the equation.

6. Solving Quadratic Equations

The standard form for a *quadratic equation* is

$$ax^2 + bx + c = 0,$$

where a, b, and c are real numbers and $a \neq 0$; for example:

$$\begin{aligned} x^2 + 6x + 5 &= 0 \\ 3x^2 - 2x &= 0, \text{ and} \\ x^2 + 4 &= 0 \end{aligned}$$

Some quadratic equations can easily be solved by factoring. For example:

$$\begin{aligned}
&(1) \qquad & x^2 + 6x + 5 &= 0 \\
&& (x+5)(x+1) &= 0 \\
&& x+5 = 0 \text{ or } x+1 &= 0 \\
&& x = -5 \text{ or } x &= -1
\end{aligned}$$

$$\begin{aligned}
&(2) \qquad & 3x^2 - 3 &= 8x \\
&& 3x^2 - 8x - 3 &= 0 \\
&& (3x+1)(x-3) &= 0 \\
&& 3x+1 = 0 \text{ or } x-3 &= 0 \\
&& x = -\frac{1}{3} \text{ or } x &= 3
\end{aligned}$$

A quadratic equation has at most two real roots and may have just one or even no real root. For example, the equation $x^2 - 6x + 9 = 0$ can be expressed as $(x - 3)^2 = 0$, or $(x - 3)(x - 3) = 0$; thus the only root is 3. The equation $x^2 + 4 = 0$ has no real root; since the square of any real number is greater than or equal to zero, $x^2 + 4$ must be greater than zero.

An expression of the form $a^2 - b^2$ can be factored as $(a - b)(a + b)$.

For example, the quadratic equation $9x^2 - 25 = 0$ can be solved as follows.

$$\begin{aligned}
(3x-5)(3x+5) &= 0 \\
3x-5 = 0 \text{ or } 3x+5 &= 0 \\
x = \frac{5}{3} \text{ or } x &= -\frac{5}{3}
\end{aligned}$$

If a quadratic expression is not easily factored, then its roots can always be found using the *quadratic formula:* If $ax^2 + bx + c = 0$ $(a \neq 0)$, then the roots are

$$x = \frac{-b + \sqrt{b^2 + 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

These are two distinct real numbers unless $b^2 - 4ac \leq 0$. If $b^2 - 4ac = 0$, then these two expressions for x are equal to $-\frac{b}{2a}$, and the equation has only one root. If $b^2 - 4ac < 0$, then $\sqrt{b^2 - 4ac}$ is not a real number and the equation has no real roots.

7. Exponents

A positive integer exponent of a number or a variable indicates a product, and the positive integer is the number of times that the number or variable is a factor in the product. For example, x^5 means $(x)(x)(x)(x)(x)$; that is, x is a factor in the product 5 times.

Some rules about exponents follow.

Let x and y be any positive numbers, and let r and s be any positive integers.

(1) $(x^r)(x^s) = x^{(r+s)}$; for example, $(2^2)(2^3) = 2^{(2+3)} = 2^5 = 32$.

(2) $\frac{x^r}{x^s} = x^{(r-s)}$; for example, $\frac{4^5}{4^2} = 4^{5-2} = 4^3 = 64$.

(3) $(x^r)(y^r) = (xy)^r$; for example, $(3^3)(4^3) = 12^3 = 1{,}728$.

(4) $\left(\frac{x}{y}\right)^r = \frac{x^r}{y^r}$; for example, $\left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3} = \frac{8}{27}$.

(5) $(x^r)^s = x^{rs} = (x^s)^r$; for example, $(x^3)^4 = x^{12} = (x^4)^3$.

(6) $x^{-r} = \frac{1}{x^r}$; for example, $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$.

(7) $x^0 = 1$; for example, $6^0 = 1$.

(8) $x^{\frac{r}{s}} = \left(x^{\frac{1}{s}}\right)^r = \left(x^r\right)^{\frac{1}{s}} = \sqrt[s]{x^r}$; for example, $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = \left(8^2\right)^{\frac{1}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$ and $9^{\frac{1}{2}} = \sqrt{9} = 3$.

It can be shown that rules 1 – 6 also apply when r and s are not integers and are not positive, that is, when r and s are any real numbers.

8. Inequalities

An *inequality* is a statement that uses one of the following symbols:

$\neq$ not equal to

$>$ greater than

$\geq$ greater than or equal to

$<$ less than

$\leq$ less than or equal to

Some examples of inequalities are $5x - 3 < 9$, $6x \geq y$, and $\frac{1}{2} < \frac{3}{4}$. Solving a linear inequality with one unknown is similar to solving an equation; the unknown is isolated on one side of the inequality. As in solving an equation, the same number can be added to or subtracted from both sides of the inequality, or both sides of an inequality can be multiplied or divided by a positive number without changing the truth of the inequality. However, multiplying or dividing an inequality by a negative number reverses the order of the inequality. For example, $6 > 2$, but $(-1)(6) < (-1)(2)$.

To solve the inequality $3x - 2 > 5$ for x, isolate x by using the following steps:

$$3x - 2 > 5$$

$$3x > 7 \quad \text{(adding 2 to both sides)}$$

$$x > \frac{7}{3} \quad \text{(dividing both sides by 3)}$$

To solve the inequality $\frac{5x-1}{-2} < 3$ for x, isolate x by using the following steps:

$$\frac{5x-1}{-2} < 3$$

$5x-1 > -6$	(multiplying both sides by -2)
$5x > -5$	(adding 1 to both sides)
$x > -1$	(dividing both sides by 5)

9. Absolute Value

The absolute value of x, denoted $|x|$, is defined to be x if $x \geq 0$ and $-x$ if $x < 0$. Note that $\sqrt{x^2}$ denotes the nonnegative square root of x^2, and so $\sqrt{x^2} = |x|$.

10. Functions

An algebraic expression in one variable can be used to define a *function* of that variable. A function is denoted by a letter such as f or g along with the variable in the expression. For example, the expression $x^3 - 5x^2 + 2$ defines a function f that can be denoted by

$$f(x) = x^3 - 5x^2 + 2.$$

The expression $\frac{2z+7}{\sqrt{z+1}}$ defines a function g that can be denoted by

$$g(z) = \frac{2z+7}{\sqrt{z+1}}.$$

The symbols "$f(x)$" or "$g(z)$" do not represent products; each is merely the symbol for an expression, and is read "f of x" or "g of z."

Function notation provides a short way of writing the result of substituting a value for a variable. If $x = 1$ is substituted in the first expression, the result can be written $f(1) = -2$, and $f(1)$ is called the "value of f at $x = 1$." Similarly, if $z = 0$ is substituted in the second expression, then the value of g at $z = 0$ is $g(0) = 7$.

Once a function $f(x)$ is defined, it is useful to think of the variable x as an input and $f(x)$ as the corresponding output. In any function there can be no more than one output for any given input. However, more than one input can give the same output; for example, if $h(x) = |x + 3|$, then $h(-4) = 1 = h(-2)$.

The set of all allowable inputs for a function is called the *domain* of the function. For f and g defined above, the domain of f is the set of all real numbers and the domain of g is the set of all numbers greater than -1. The domain of any function can be arbitrarily specified, as in the function defined by "$h(x) = 9x - 5$ for $0 \leq x \leq 10$." Without such a restriction, the domain is assumed to be all values of x that result in a real number when substituted into the function.

The domain of a function can consist of only the positive integers and possibly 0. For example, $a(n) = n^2 + \frac{n}{5}$ for $n = 0, 1, 2, 3, \ldots$ . .

Such a function is called a *sequence* and $a(n)$ is denoted by a_n. The value of the sequence a_n at $n = 3$ is $a_3 = 3^2 + \frac{3}{5} = 9.60$. As another example, consider the sequence defined by $b_n = (-1)^n (n!)$ for $n = 1, 2, 3, \ldots$ A sequence like this is often indicated by listing its values in the order $b_1, b_2, b_3, \ldots, b_n, \ldots$ as follows:

$-1, 2, -6, \ldots, (-1)^n(n!), \ldots$, and $(-1)^n(n!)$ is called the nth term of the sequence.

3.3 Geometry

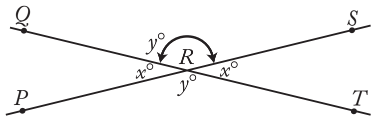
1. Lines

In geometry, the word "line" refers to a straight line that extends without end in both directions.

The line above can be referred to as line PQ or line ℓ. The part of the line from P to Q is called a *line segment*. P and Q are the *endpoints* of the segment. The notation $\overline{PQ}$ is used to denote line segment PQ and PQ is used to denote the length of the segment.

2. Intersecting Lines and Angles

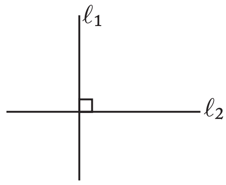
If two lines intersect, the opposite angles are called *vertical angles* and have the same measure. In the figure



$\angle PRQ$ and $\angle SRT$ are vertical angles and $\angle QRS$ and $\angle PRT$ are vertical angles. Also, $x + y = 180°$ since PRS is a straight line.

3. Perpendicular Lines

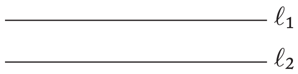
An angle that has a measure of 90° is a *right angle*. If two lines intersect at right angles, the lines are *perpendicular*. For example:



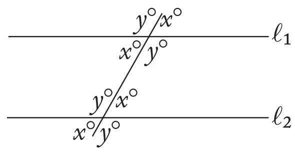
ℓ_1 and ℓ_2 above are perpendicular, denoted by $\ell_1 \perp \ell_2$. A right angle symbol in an angle of intersection indicates that the lines are perpendicular.

4. Parallel Lines

If two lines that are in the same plane do not intersect, the two lines are *parallel*. In the figure



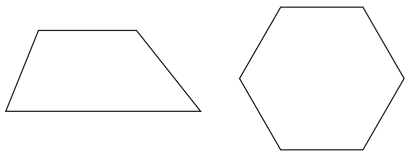
lines ℓ_1 and ℓ_2 are parallel, denoted by $\ell_1 \parallel \ell_2$. If two parallel lines are intersected by a third line, as shown below, then the angle measures are related as indicated, where $x + y = 180°$.



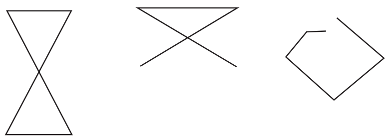
5. Polygons (Convex)

A *polygon* is a closed plane figure formed by three or more line segments, called the *sides* of the polygon. Each side intersects exactly two other sides at their endpoints. The points of intersection of the sides are *vertices*. The term "polygon" will be used to mean a convex polygon, that is, a polygon in which each interior angle has a measure of less than 180°.

The following figures are polygons:

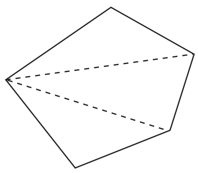


The following figures are not polygons:



A polygon with three sides is a *triangle*; with four sides, a *quadrilateral*; with five sides, a *pentagon*; and with six sides, a *hexagon*.

The sum of the interior angle measures of a triangle is 180°. In general, the sum of the interior angle measures of a polygon with n sides is equal to $(n - 2)180°$. For example, this sum for a pentagon is $(5 - 2)180° = (3)180° = 540°$.



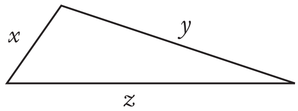
Note that a pentagon can be partitioned into three triangles and therefore the sum of the angle measures can be found by adding the sum of the angle measures of three triangles.

The *perimeter* of a polygon is the sum of the lengths of its sides.

The commonly used phrase "area of a triangle" (or any other plane figure) is used to mean the area of the region enclosed by that figure.

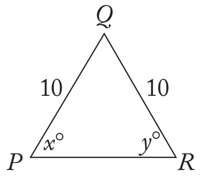
6. Triangles

There are several special types of triangles with important properties. But one property that all triangles share is that the sum of the lengths of any two of the sides is greater than the length of the third side, as illustrated below.

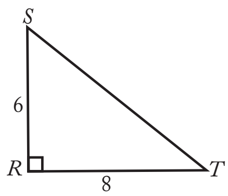


$x + y > z$, $x + z > y$, and $y + z > x$

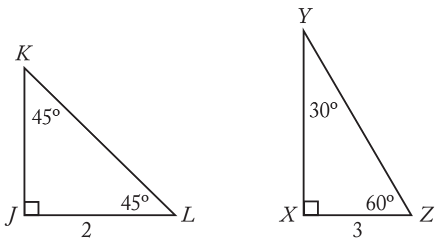
An *equilateral* triangle has all sides of equal length. All angles of an equilateral triangle have equal measure. An *isosceles* triangle has at least two sides of the same length. If two sides of a triangle have the same length, then the two angles opposite those sides have the same measure. Conversely, if two angles of a triangle have the same measure, then the sides opposite those angles have the same length. In isosceles triangle PQR below, $x = y$ since $PQ = QR$.



A triangle that has a right angle is a *right* triangle. In a right triangle, the side opposite the right angle is the *hypotenuse,* and the other two sides are the *legs.* An important theorem concerning right triangles is the *Pythagorean theorem,* which states: In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.



In the figure above, ΔRST is a right triangle, so $(RS)^2 + (RT)^2 = (ST)^2$. Here, $RS = 6$ and $RT = 8$, so $ST = 10$, since $6^2 + 8^2 = 36 + 64 = 100 = (ST)^2$ and $ST = \sqrt{100}$. Any triangle in which the lengths of the sides are in the ratio 3:4:5 is a right triangle. In general, if a, b, and c are the lengths of the sides of a triangle and $a^2 + b^2 = c^2$, then the triangle is a right triangle.

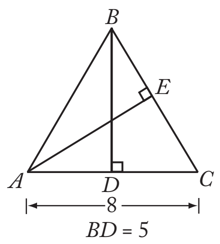


In 45°– 45°– 90° triangles, the lengths of the sides are in the ratio $1:1:\sqrt{2}$. For example, in ΔJKL, if $JL = 2$, then $JK = 2$ and $KL = 2\sqrt{2}$. In 30°– 60°– 90° triangles, the lengths of the sides are in the ratio $1:\sqrt{3}:2$. For example, in ΔXYZ, if $XZ = 3$, then $XY = 3\sqrt{3}$ and $YZ = 6$.

The *altitude* of a triangle is the segment drawn from a vertex perpendicular to the side opposite that vertex. Relative to that vertex and altitude, the opposite side is called the *base.*

The area of a triangle is equal to:

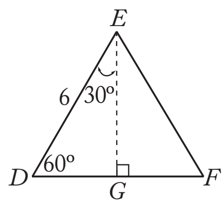
$$\frac{(\text{the length of the altitude}) \times (\text{the length of the base})}{2}$$



In ΔABC, $\overline{BD}$ is the altitude to base $\overline{AC}$ and $\overline{AE}$ is the altitude to base $\overline{BC}$. The area of ΔABC is equal to

$$\frac{BD \times AC}{2} = \frac{5 \times 8}{2} = 20.$$

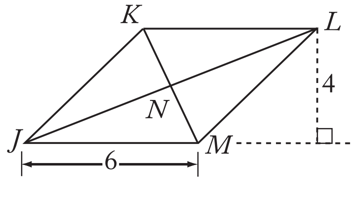
The area is also equal to $\frac{AE \times BC}{2}$. If ΔABC above is isosceles and $AB = BC$, then altitude $\overline{BD}$ bisects the base; that is, $AD = DC = 4$. Similarly, any altitude of an equilateral triangle bisects the side to which it is drawn.



In equilateral triangle *DEF*, if $DE = 6$, then $DG = 3$ and $EG = 3\sqrt{3}$. The area of ΔDEF is equal to $\frac{3\sqrt{3} \times 6}{2} = 9\sqrt{3}$.

7. Quadrilaterals

A polygon with four sides is a *quadrilateral.* A quadrilateral in which both pairs of opposite sides are parallel is a *parallelogram.* The opposite sides of a parallelogram also have equal length.



In parallelogram *JKLM*, $\overline{JK} \parallel \overline{LM}$ and $JK = LM$; $\overline{KL} \parallel \overline{JM}$ and $KL = JM$.

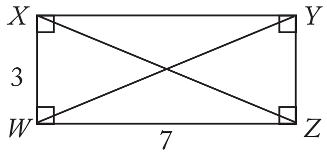
The diagonals of a parallelogram bisect each other (that is, $KN = NM$ and $JN = NL$).

The area of a parallelogram is equal to

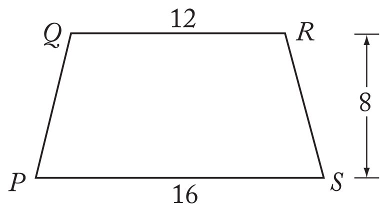
(the length of the altitude) × (the length of the base).

The area of *JKLM* is equal to $4 \times 6 = 24$.

A parallelogram with right angles is a *rectangle,* and a rectangle with all sides of equal length is a *square.*



The perimeter of $WXYZ = 2(3) + 2(7) = 20$ and the area of *WXYZ* is equal to $3 \times 7 = 21$. The diagonals of a rectangle are equal; therefore $WY = XZ = \sqrt{9+49} = \sqrt{58}$.



A quadrilateral with two sides that are parallel, as shown above, is a *trapezoid.* The area of trapezoid *PQRS* may be calculated as follows:

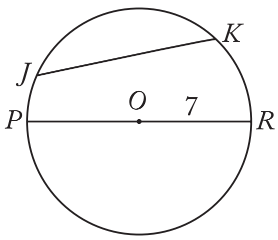
$$\frac{1}{2}(\text{the sum of the lengths of the bases})(\text{the height}) = \frac{1}{2}(QR + PS)(8) = \frac{1}{2}(28 \times 8) = 112.$$

8. Circles

A *circle* is a set of points in a plane that are all located the same distance from a fixed point (the *center* of the circle).

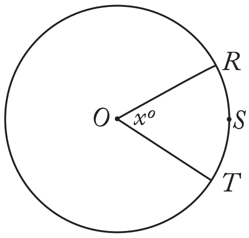
A *chord* of a circle is a line segment that has its endpoints on the circle. A chord that passes through the center of the circle is a *diameter* of the circle. A *radius* of a circle is a segment from the center of the circle to a point on the circle. The words "diameter" and "radius" are also used to refer to the lengths of these segments.

The *circumference* of a circle is the distance around the circle. If r is the radius of the circle, then the circumference is equal to $2\pi r$, where π is approximately $\frac{22}{7}$ or 3.14. The *area* of a circle of radius r is equal to πr^2.



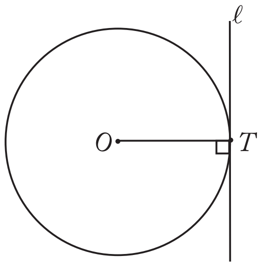
In the circle above, O is the center of the circle and $\overline{JK}$ and $\overline{PR}$ are chords. $\overline{PR}$ is a diameter and $\overline{OR}$ is a radius. If $OR = 7$, then the circumference of the circle is $2\pi(7) = 14\pi$ and the area of the circle is $\pi(7)^2 = 49\pi$.

The number of degrees of arc in a circle (or the number of degrees in a complete revolution) is 360.



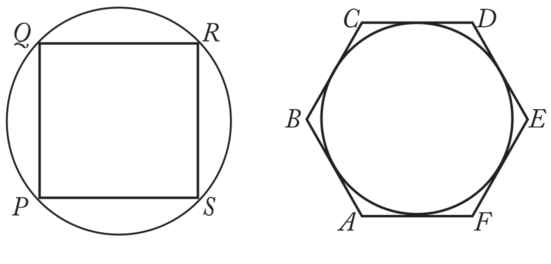
In the circle with center O above, the length of arc RST is $\frac{x}{360}$ of the circumference of the circle; for example, if $x = 60$, then arc RST has length $\frac{1}{6}$ of the circumference of the circle.

A line that has exactly one point in common with a circle is said to be *tangent* to the circle, and that common point is called the *point of tangency.* A radius or diameter with an endpoint at the point of tangency is perpendicular to the tangent line, and, conversely, a line that is perpendicular to a radius or diameter at one of its endpoints is tangent to the circle at that endpoint.



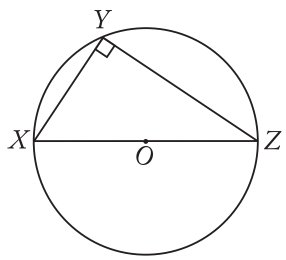
The line ℓ above is tangent to the circle and radius $\overline{OT}$ is perpendicular to ℓ.

If each vertex of a polygon lies on a circle, then the polygon is *inscribed* in the circle and the circle is *circumscribed* about the polygon. If each side of a polygon is tangent to a circle, then the polygon is *circumscribed* about the circle and the circle is *inscribed* in the polygon.



In the figure above, quadrilateral $PQRS$ is inscribed in a circle and hexagon $ABCDEF$ is circumscribed about a circle.

If a triangle is inscribed in a circle so that one of its sides is a diameter of the circle, then the triangle is a right triangle.



In the circle above, $\overline{XZ}$ is a diameter and the measure of $\angle XYZ$ is 90°.

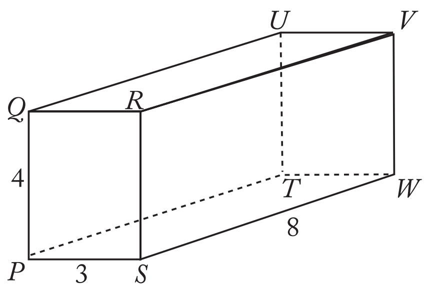
9. Rectangular Solids and Cylinders

A *rectangular solid* is a three-dimensional figure formed by 6 rectangular surfaces, as shown below. Each rectangular surface is a *face*. Each solid or dotted line segment is an *edge,* and each point at which the edges meet is a *vertex.* A rectangular solid has 6 faces, 12 edges, and 8 vertices. Opposite faces are parallel rectangles that have the same dimensions. A rectangular solid in which all edges are of equal length is a *cube*.

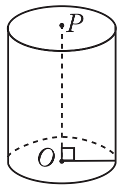
The *surface area* of a rectangular solid is equal to the sum of the areas of all the faces. The *volume* is equal to

$$(\text{length}) \times (\text{width}) \times (\text{height});$$

$$\text{in other words, } (\text{area of base}) \times (\text{height}).$$



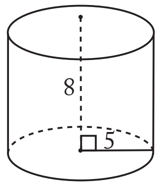
In the rectangular solid above, the dimensions are 3, 4, and 8. The surface area is equal to $2(3 \times 4) + 2(3 \times 8) + 2(4 \times 8) = 136$. The volume is equal to $3 \times 4 \times 8 = 96$.



The figure above is a right circular *cylinder*. The two bases are circles of the same size with centers O and P, respectively, and altitude (height) $\overline{OP}$ is perpendicular to the bases. The surface area of a right circular cylinder with a base of radius r and height h is equal to $2(\pi r^2) + 2\pi rh$ (the sum of the areas of the two bases plus the area of the curved surface).

The volume of a cylinder is equal to $\pi r^2 h$, that is,

$$\text{(area of base)} \times \text{(height)}.$$



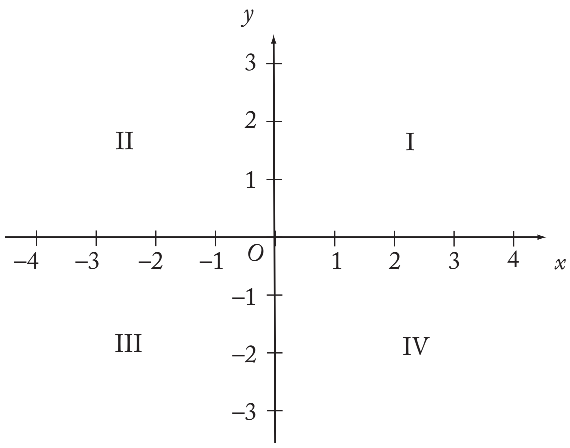
In the cylinder above, the surface area is equal to

$$2(25\pi) + 2\pi(5)(8) = 130\pi,$$

and the volume is equal to

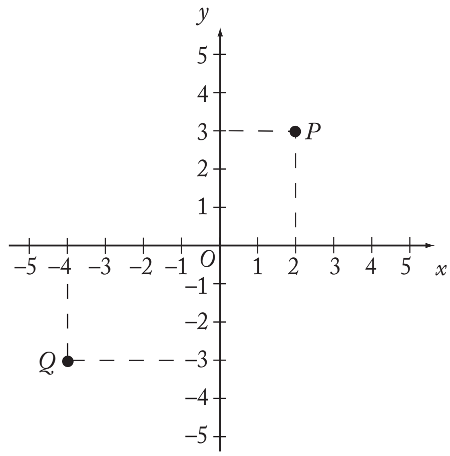
$$25\pi(8) = 200\pi.$$

10. Coordinate Geometry



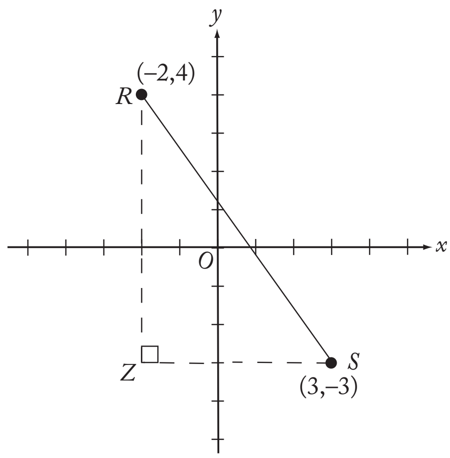
The figure above shows the (rectangular) *coordinate plane*. The horizontal line is called the *x-axis* and the perpendicular vertical line is called the *y-axis*. The point at which these two axes intersect, designated O, is called the *origin*. The axes divide the plane into four quadrants, I, II, III, and IV, as shown.

Each point in the plane has an *x-coordinate* and a *y-coordinate.* A point is identified by an ordered pair (x,y) of numbers in which the x-coordinate is the first number and the y-coordinate is the second number.



In the graph above, the (x,y) coordinates of point P are (2,3) since P is 2 units to the right of the y-axis (that is, $x = 2$) and 3 units above the x-axis (that is, $y = 3$). Similarly, the (x,y) coordinates of point Q are (−4,−3). The origin O has coordinates (0,0).

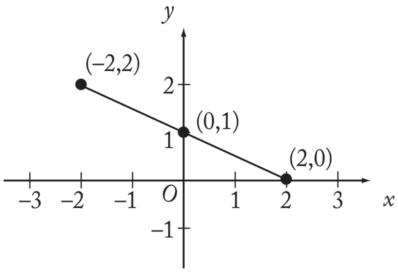
One way to find the distance between two points in the coordinate plane is to use the Pythagorean theorem.



To find the distance between points R and S using the Pythagorean theorem, draw the triangle as shown. Note that Z has (x,y) coordinates (−2,−3), $RZ = 7$, and $ZS = 5$. Therefore, the distance between R and S is equal to

$$\sqrt{7^2 + 5^2} = \sqrt{74}.$$

For a line in the coordinate plane, the coordinates of each point on the line satisfy a linear equation of the form $y = mx + b$ (or the form $x = a$ if the line is vertical). For example, each point on the line on the next page satisfies the equation $y = -\frac{1}{2}x + 1$. One can verify this for the points (−2,2), (2,0), and (0,1) by substituting the respective coordinates for x and y in the equation.



In the equation $y = mx + b$ of a line, the coefficient m is the *slope* of the line and the constant term b is the y-intercept of the line. For any two points on the line, the slope is defined to be the ratio of the difference in the y-coordinates to the difference in the x-coordinates. Using (–2, 2) and (2, 0) above, the slope is

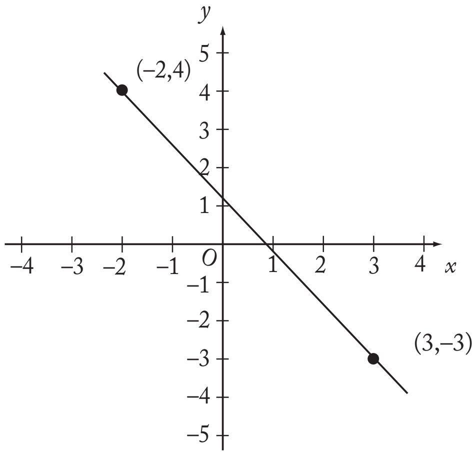
$$\frac{\text{The difference in the } y\text{-coordinates}}{\text{The difference in the } x\text{-coordinates}} = \frac{0-2}{2-(-2)} = \frac{-2}{4} = -\frac{1}{2}.$$

The y-intercept is the y-coordinate of the point at which the line intersects the y-axis. For the line above, the y-intercept is 1, and this is the resulting value of y when x is set equal to 0 in the equation $y = -\frac{1}{2}x + 1$. The x-intercept is the x-coordinate of the point at which the line intersects the x-axis. The x-intercept can be found by setting $y = 0$ and solving for x. For the line $y = -\frac{1}{2}x + 1$, this gives

$$-\frac{1}{2}x + 1 = 0$$
$$-\frac{1}{2}x = -1$$
$$x = 2.$$

Thus, the x-intercept is 2.

Given any two points (x_1,y_1) and (x_2,y_2) with $x_1 \neq x_2$, the equation of the line passing through these points can be found by applying the definition of slope. Since the slope is $m = \frac{y_2 - y_1}{x_2 - x_1}$, then using a point known to be on the line, say (x_1,y_1), any point (x,y) on the line must satisfy $\frac{y - y_1}{x - x_1} = m$, or $y - y_1 = m(x - x_1)$. (Using (x_2,y_2) as the known point would yield an equivalent equation.) For example, consider the points (–2,4) and (3,–3) on the line below.



The slope of this line is $\frac{-3-4}{3-(-2)}=\frac{-7}{5}$, so an equation of this line can be found using the point (3,–3) as follows:

$$y-(-3)=-\frac{7}{5}(x-3)$$
$$y+3=-\frac{7}{5}x+\frac{21}{5}$$
$$y=-\frac{7}{5}x+\frac{6}{5}$$

The y-intercept is $\frac{6}{5}$. The x-intercept can be found as follows:

$$0=-\frac{7}{5}x+\frac{6}{5}$$
$$\frac{7}{5}x=\frac{6}{5}$$
$$x=\frac{6}{7}$$

Both of these intercepts can be seen on the graph.

If the slope of a line is negative, the line slants downward from left to right; if the slope is positive, the line slants upward. If the slope is 0, the line is horizontal; the equation of such a line is of the form $y = b$ since $m = 0$. For a vertical line, slope is not defined, and the equation is of the form $x = a$, where a is the x-intercept.

There is a connection between graphs of lines in the coordinate plane and solutions of two linear equations with two unknowns. If two linear equations with unknowns x and y have a unique solution, then the graphs of the equations are two lines that intersect in one point, which is the solution. If the equations are equivalent, then they represent the same line with infinitely many points or solutions. If the equations have no solution, then they represent parallel lines, which do not intersect.

There is also a connection between functions (see section 3.2.10) and the coordinate plane. If a function is graphed in the coordinate plane, the function can be understood in different and useful ways. Consider the function defined by

$$f(x) = -\frac{7}{5}x + \frac{6}{5}.$$

If the value of the function, $f(x)$, is equated with the variable y, then the graph of the function in the xy-coordinate plane is simply the graph of the equation

$$y = -\frac{7}{5}x + \frac{6}{5}$$

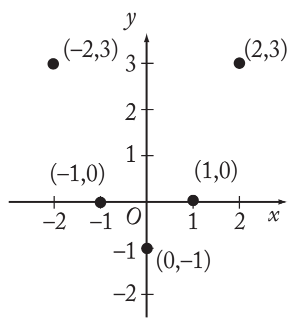
shown above. Similarly, any function $f(x)$ can be graphed by equating y with the value of the function:

$$y = f(x).$$

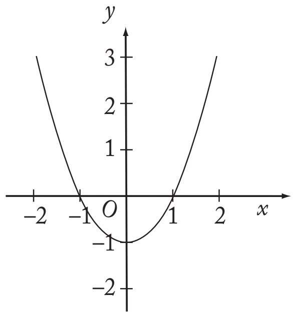
So for any x in the domain of the function f, the point with coordinates $(x, f(x))$ is on the graph of f, and the graph consists entirely of these points.

As another example, consider a quadratic polynomial function defined by $f(x) = x^2 - 1$. One can plot several points $(x, f(x))$ on the graph to understand the connection between a function and its graph:

x	$f(x)$
–2	3
–1	0
0	–1
1	0
2	3



If all the points were graphed for $-2 \leq x \leq 2$, then the graph would appear as follows.



The graph of a quadratic function is called a *parabola* and always has the shape of the curve above, although it may be upside down or have a greater or lesser width. Note that the roots of the equation $f(x) = x^2 - 1 = 0$ are $x = 1$ and $x = -1$; these coincide with the x-intercepts since x-intercepts are found by setting $y = 0$ and solving for x. Also, the y-intercept is $f(0) = -1$ because this is the value of y corresponding to $x = 0$. For any function f, the x-intercepts are the solutions of the equation $f(x) = 0$ and the y-intercept is the value $f(0)$.

3.4 Word Problems

Many of the principles discussed in this chapter are used to solve word problems. The following discussion of word problems illustrates some of the techniques and concepts used in solving such problems.

1. Rate Problems

The distance that an object travels is equal to the product of the average speed at which it travels and the amount of time it takes to travel that distance, that is,

$$\text{Rate} \times \text{Time} = \text{Distance}.$$

Example 1: If a car travels at an average speed of 70 kilometers per hour for 4 hours, how many kilometers does it travel?

Solution: Since rate × time = distance, simply multiply 70 km/hour × 4 hours. Thus, the car travels 280 kilometers in 4 hours.

To determine the average rate at which an object travels, divide the total distance traveled by the total amount of traveling time.

Example 2: On a 400-mile trip, Car X traveled half the distance at 40 miles per hour (mph) and the other half at 50 mph. What was the average speed of Car X ?

Solution: First it is necessary to determine the amount of traveling time. During the first 200 miles, the car traveled at 40 mph; therefore, it took $\frac{200}{40} = 5$ hours to travel the first 200 miles.

During the second 200 miles, the car traveled at 50 mph; therefore, it took $\frac{200}{50} = 4$ hours to travel the second 200 miles. Thus, the average speed of Car X was $\frac{400}{9} = 44\frac{4}{9}$ mph. Note that the average speed is *not* $\frac{40+50}{2} = 45$.

Some rate problems can be solved by using ratios.

Example 3: If 5 shirts cost $44, then, at this rate, what is the cost of 8 shirts?

Solution: If c is the cost of the 8 shirts, then $\frac{5}{44}=\frac{8}{c}$. Cross multiplication results in the equation

$$5c = 8 \times 44 = 352$$

$$c = \frac{352}{5} = 70.40$$

The 8 shirts cost $70.40.

2. Work Problems

In a work problem, the rates at which certain persons or machines work alone are usually given, and it is necessary to compute the rate at which they work together (or vice versa).

The basic formula for solving work problems is $\frac{1}{r}+\frac{1}{s}=\frac{1}{h}$, where r and s are, for example, the number of hours it takes Rae and Sam, respectively, to complete a job when working alone, and h is the number of hours it takes Rae and Sam to do the job when working together. The reasoning is that in 1 hour Rae does $\frac{1}{r}$ of the job, Sam does $\frac{1}{s}$ of the job, and Rae and Sam together do $\frac{1}{h}$ of the job.

Example 1: If Machine X can produce 1,000 bolts in 4 hours and Machine Y can produce 1,000 bolts in 5 hours, in how many hours can Machines X and Y, working together at these constant rates, produce 1,000 bolts?

Solution:

$$\frac{1}{4}+\frac{1}{5}=\frac{1}{h}$$

$$\frac{5}{20}+\frac{4}{20}=\frac{1}{h}$$

$$\frac{9}{20}=\frac{1}{h}$$

$$9h = 20$$

$$h = \frac{20}{9} = 2\frac{2}{9}$$

Working together, Machines X and Y can produce 1,000 bolts in $2\frac{2}{9}$ hours.

Example 2: If Art and Rita can do a job in 4 hours when working together at their respective constant rates and Art can do the job alone in 6 hours, in how many hours can Rita do the job alone?

Solution:

$$\frac{1}{6}+\frac{1}{R}=\frac{1}{4}$$
$$\frac{R+6}{6R}=\frac{1}{4}$$
$$4R+24=6R$$
$$24=2R$$
$$12=R$$

Working alone, Rita can do the job in 12 hours.

3. Mixture Problems

In mixture problems, substances with different characteristics are combined, and it is necessary to determine the characteristics of the resulting mixture.

Example 1: If 6 pounds of nuts that cost \$1.20 per pound are mixed with 2 pounds of nuts that cost \$1.60 per pound, what is the cost per pound of the mixture?

Solution: The total cost of the 8 pounds of nuts is

$$6(\$1.20)+2(\$1.60)=\$10.40.$$

The cost per pound is $$\frac{\$10.40}{8}=\$1.30.$$

Example 2: How many liters of a solution that is 15 percent salt must be added to 5 liters of a solution that is 8 percent salt so that the resulting solution is 10 percent salt?

Solution: Let n represent the number of liters of the 15% solution. The amount of salt in the 15% solution $[0.15n]$ plus the amount of salt in the 8% solution $[(0.08)(5)]$ must be equal to the amount of salt in the 10% mixture $[0.10(n + 5)]$. Therefore,

$$0.15n+0.08(5)=0.10(n+5)$$
$$15n+40=10n+50$$
$$5n=10$$
$$n=2 \text{ liters}$$

Two liters of the 15% salt solution must be added to the 8% solution to obtain the 10% solution.

4. Interest Problems

Interest can be computed in two basic ways. With simple annual interest, the interest is computed on the principal only and is equal to (principal) × (interest rate) × (time). If interest is compounded, then interest is computed on the principal as well as on any interest already earned.

Example 1: If \$8,000 is invested at 6 percent simple annual interest, how much interest is earned after 3 months?

Solution: Since the annual interest rate is 6%, the interest for 1 year is

$$(0.06)(\$8,000) = \$480.$$

The interest earned in 3 months is $\frac{3}{12}(\$480) = \$120.$

Example 2: If \$10,000 is invested at 10 percent annual interest, compounded semiannually, what is the balance after 1 year?

Solution: The balance after the first 6 months would be

$$10,000 + (10,000)(0.05) = \$10,500.$$

The balance after one year would be $10,500 + (10,500)(0.05) = \$11,025.$

Note that the interest rate for each 6-month period is 5%, which is half of the 10% annual rate. The balance after one year can also be expressed as

$$10,000\left(1+\frac{0.10}{2}\right)^2 \text{ dollars.}$$

5. Discount

If a price is discounted by n percent, then the price becomes $(100 - n)$ percent of the original price.

Example 1: A certain customer paid \$24 for a dress. If that price represented a 25 percent discount on the original price of the dress, what was the original price of the dress?

Solution: If p is the original price of the dress, then $0.75p$ is the discounted price and $0.75p = \$24$, or $p = \$32$. The original price of the dress was \$32.

Example 2: The price of an item is discounted by 20 percent and then this reduced price is discounted by an additional 30 percent. These two discounts are equal to an overall discount of what percent?

Solution: If p is the original price of the item, then $0.8p$ is the price after the first discount. The price after the second discount is $(0.7)(0.8)\,p = 0.56p$. This represents an overall discount of 44 percent (100% – 56%).

6. Profit

Gross profit is equal to revenues minus expenses, or selling price minus cost.

Example: A certain appliance costs a merchant \$30. At what price should the merchant sell the appliance in order to make a gross profit of 50 percent of the cost of the appliance?

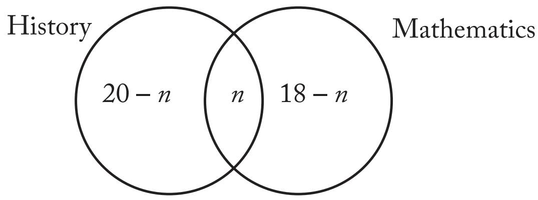
Solution: If s is the selling price of the appliance, then $s - 30 = (0.5)(30)$, or $s = \$45$. The merchant should sell the appliance for \$45.

7. Sets

If S is the set of numbers 1, 2, 3, and 4, you can write $S = \{1, 2, 3, 4\}$. Sets can also be represented by Venn diagrams. That is, the relationship among the members of sets can be represented by circles.

Example 1: Each of 25 people is enrolled in history, mathematics, or both. If 20 are enrolled in history and 18 are enrolled in mathematics, how many are enrolled in both history and mathematics?

Solution: The 25 people can be divided into three sets: those who study history only, those who study mathematics only, and those who study history and mathematics. Thus a Venn diagram may be drawn as follows, where n is the number of people enrolled in both courses, $20 - n$ is the number enrolled in history only, and $18 - n$ is the number enrolled in mathematics only.



Since there is a total of 25 people, $(20 - n) + n + (18 - n) = 25$, or $n = 13$. Thirteen people are enrolled in both history and mathematics. Note that $20 + 18 - 13 = 25$, which is the general addition rule for two sets (see section 3.1.9).

Example 2: In a certain production lot, 40 percent of the toys are red and the remaining toys are green. Half of the toys are small and half are large. If 10 percent of the toys are red and small, and 40 toys are green and large, how many of the toys are red and large?

Solution: For this kind of problem, it is helpful to organize the information in a table:

	Red	Green	Total
Small	10%		50%
Large			50%
Total	40%	60%	100%

The numbers in the table are the percentages given. The following percentages can be computed on the basis of what is given:

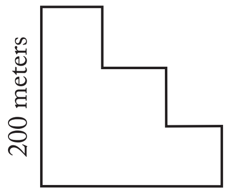
	Red	Green	Total
Small	10%	40%	50%
Large	30%	20%	50%
Total	40%	60%	100%

Since 20% of the number of toys (n) are green and large, $0.20n = 40$ (40 toys are green and large), or $n = 200$. Therefore, 30% of the 200 toys, or $(0.3)(200) = 60$, are red and large.

8. Geometry Problems

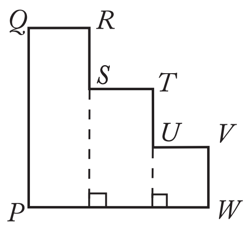
The following is an example of a word problem involving geometry.

Example:



The figure above shows an aerial view of a piece of land. If all angles shown are right angles, what is the perimeter of the piece of land?

Solution: For reference, label the figure as



If all the angles are right angles, then $QR + ST + UV = PW$, and $RS + TU + VW = PQ$. Hence, the perimeter of the land is $2PW + 2PQ = 2 \times 200 + 2 \times 200 = 800$ meters.

9. Measurement Problems

Some questions on the GMAT involve metric units of measure, whereas others involve English units of measure. However, except for units of time, if a question requires conversion from one unit of measure to another, the relationship between those units will be given.

Example: A train travels at a constant rate of 25 meters per second. How many kilometers does it travel in 5 minutes? (1 kilometer = 1,000 meters)

Solution: In 1 minute the train travels (25)(60) = 1,500 meters, so in 5 minutes it travels 7,500 meters. Since 1 kilometer = 1,000 meters, it follows that 7,500 meters equals $\frac{7,500}{1,000}$, or 7.5 kilometers.